Equivariant multiplicities of Mirković-Vilonen cycles and Weyl group action

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Motivation

$G \supset B \supset T$, W, $N = \text{unip. rad}(B)$, $\underline{t} = \text{Lie}\, T$
quasi simple $/\mathbb{C}$

$\mathfrak{g} = \text{Lie}\, G \supset \underset{\text{nilpotent cone}}{\mathcal{N}} \supset \underline{n} = \text{Lie}\, N$

$\pi : G \overset{B}{\times} \underline{n} \longrightarrow \mathcal{N}$ Springer resolution

$u \in \mathcal{N} \rightsquigarrow$ $\mathcal{O}_u$ nilpotent orbit

($\pi^{-1}u = \mathcal{B}_u$ Springer fiber, of pure dimension $\frac{1}{2}\operatorname{codim}_{\mathcal{N}}(\mathcal{O}_u)$)

$W \circlearrowright H^{top}(\pi^{-1}u)^{G_u}$ ← (isotropy subgroup of u) Springer's representation σ_u

$$\begin{array}{ccccc} \operatorname{Irr}(\overline{\mathcal{O}_u \cap \underline{n}}) & \overset{1:1}{\longleftrightarrow} & \operatorname{Irr}(G \overset{B}{\times} (\overline{\mathcal{O}_u \cap \underline{n}})) & \overset{1:1}{\longleftrightarrow} & \operatorname{Irr}(\pi^{-1}u)/G_u \\ \text{orbital varieties} & & & & \\ C \subset \overline{\mathcal{O}_u \cap n} & \longmapsto & G \overset{B}{\times} C & \longmapsto & \operatorname{Irr}((G \overset{B}{\times} C) \cap \pi^{-1}u) \end{array}$$

Joseph. C orbital variety $\mapsto p_C \in \mathbb{C}[\underline{t}]$ characteristic polynomial

encodes the asymptotics of weight multiplicities in the T-module $\mathbb{C}[C]$

(weights of $\mathbb{C}[C]$ are ≥ 0; applying a dominant $\gamma \in X_*(T)$ defines a $\mathbb{N}$-graduation on $\mathbb{C}[C] \rightsquigarrow p_C(\gamma)$ is the leading coeff. of the Hilbert polynomial, that is, a multiplicity.)

Hotta: $\operatorname{Span}\{p_C \mid C \in \operatorname{Irr}(\overline{\mathcal{O}_u \cap \underline{n}})\} \subset \mathbb{C}[\underline{t}]$ W-module isomorphic to σ_u.

Borho-Brylinski-MacPherson: relation with equivariant K-theory and equivariant cohomology.

Equivariant multiplicities

scheme $X \circlearrowleft T$ torus, $\underline{t} = \mathrm{Lie}\, T$, $S = S_{\mathbb{Z}} X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[\underline{t}_{\mathbb{Q}}]$

Assume: X^T finite, and T acts with non-zero weights on each $T_x X$, $x \in X^T$.

(eg X embeds T-equivariantly in a smooth variety with finitely many fixed points)

$\rightsquigarrow$ equivariant multiplicity $e_x^T(X) \in \mathrm{Frac}(S) = \mathbb{Q}(\underline{t}_{\mathbb{Q}})$ at each fixed point $x \in X^T$ (Brion)

example: X smooth at x: $e_x^T(X) = \dfrac{1}{\chi_1 \cdots \chi_n}$ where $\chi_1, .., \chi_n$ = weights of T acting on $T_x X$

Prop: $i: X^T \hookrightarrow X$,

localization theorem in equivariant homology: $i_*: H_\bullet^T(X^T) \otimes_S \mathrm{Frac}(S) \xrightarrow{\simeq} H_\bullet^T(X) \otimes_S \mathrm{Frac}(S)$

$$\sum_{x \in X^T} e_x^T(X) \cdot [x] \longmapsto [X]$$

Then, for $C \in \mathrm{Irr}(\overline{\mathbb{O}_u \cap \underline{n}})$, $e_0^T(C) = \dfrac{P_c}{\prod\limits_{\substack{\alpha \text{ root} \\ \alpha > 0}} \alpha}$,

Geometric Satake

$\mathcal{O} = \mathbb{C}[[t]]$, $\mathcal{K} = \mathbb{C}((t))$

$Gr = G(\mathcal{K})/G(\mathcal{O})$

$\lambda \in X_*(T) \rightsquigarrow t^\lambda \in T(\mathcal{K}) \rightsquigarrow L_\lambda \in Gr$, $\quad Gr^\lambda = G(\mathcal{O}) \cdot L_\lambda \subset Gr$

λ dominant $\rightsquigarrow$ $L(\lambda)$ irreducible repr. of $G^\vee$ (Langlands dual) with highest weight λ.

Geometric Satake (Lusztig, Beilinson-Drinfeld, Mirković-Vilonen): $L(\lambda) = IH(\overline{Gr^\lambda}, \mathbb{C})$

Weight spaces: $\mu \in X_*(T) \rightsquigarrow S_\mu^- = N^-(\mathcal{K}) \cdot L_\mu \subset Gr$

$$L(\lambda)_\mu = \underbrace{IH^{2\rho^\vee(\mu)}_{S_\mu^-}}_{\text{supported on } S_\mu^-}(\overline{Gr^\lambda}, \mathbb{C}) = H_{top}(\overline{Gr^\lambda} \cap S_\mu^-, \mathbb{C})$$

$$Z \in \mathrm{Irr}(\overline{Gr^\lambda} \cap S_\mu^-) \mapsto v_Z \in L(\lambda)_\mu \text{ basis vector}$$

Conjecture (Muthiah, '18): The map $L(\lambda)_0 \to \mathbb{C}(\underline{t})$ that linearly extends $v_Z \mapsto e_0^T(Z)$ is W-equivariant.

Theorem (Muthiah): True for $G^\vee = SL_d$ and $\lambda \leqslant d\varpi_1$ (i.e. $L(\lambda) \subset (\mathbb{C}^d)^{\otimes d}$)

(Deduced from Joseph and Hotta's result by Schur-Weyl duality. The comparison on the geometric side is rather delicate, since MV cycles $\neq$ orbital varieties!)

Theorem (BKK): True in general.

(Difficulty: don't know how W acts through Geometric Satake; don't know much about the geometry of MV cycles.)

Sketch of proof

1) Atiyah–Bott's integration formula $\underbrace{\int_{x\in X} e^{\langle m(x),y\rangle}\, d\mathrm{vol}}$ $= \sum_{x\in X^T} e^T_x(X)\, e^{\langle m(x),y\rangle}$

here, X compact symplectic manifold
$\circlearrowleft$
T hamiltonian action of compact torus with moment map $m: X \to \underline{t}^*$

value at $y \in \underline{t}_{\mathbb{C}}$ of Fourier–Laplace transf. of Duistermaat–Heckman measure of X

recover the equiv. multiplicities here

Still valid for X algebraic projective (possibly non-smooth), but need an alternate definition of DH measure (Brion–Procesi).

(Proof of integration formula then involves different ingredients: localisation in equivariant K-theory.)

2) Back to Geometric Satake

$\{\alpha_i^\vee \mid i\in I\}$ simple roots of $\mathfrak{g}^\vee$, pick $e_i \in \underline{n}^\vee$ simple root vectors, $e = \sum_{i\in I} e_i$ principal nilpotent.

$\mathcal{M} = \{$complex measures on $\underline{t}_{\mathbb{R}}$ w/ compact support$\}$

$\Delta_n = \{(s_1,\dots,s_n)\in\mathbb{R}^n \mid 1 \geq s_1 \geq \dots \geq s_n \geq 0\}$ standard simplex

$\underline{i} = (i_1,\dots,i_n) \rightsquigarrow$ monomial $e_{\underline{i}} = e_{i_1}\cdots e_{i_n} \in U(\underline{n}^\vee)$

measure $D_{\underline{i}} \in \mathcal{M}$, image of Lebesgue by $\Delta_n \to \underline{t}_{\mathbb{R}}$, $(s_1,\dots,s_n) \mapsto -(s_1\alpha_{i_1}^\vee + \dots + s_n\alpha_{i_n}^\vee)$

Take $q: X_*(T) \to \mathbb{Q}$ W-invariant quadratic form

$\rightsquigarrow$ T-equivariant projective embedding of $\mathcal{G}r$ (first define central extension of $G(\mathcal{K})$)

$\iota: X_*(T) \to X^*(T)\otimes_{\mathbb{Z}}\mathbb{Q}$ polar form (moment map on the fixed points for the rhs: $m(L_\lambda) = \iota(\lambda)$)

Proposition: $\lambda,\mu \in X_*(T)$, λ dominant, $Z \in \mathrm{Irr}(\mathcal{G}r^\lambda \cap S_\mu^-)$, $v_Z \in L(\lambda)_\mu \cong H_{\mathrm{top}}(\overline{\mathcal{G}r^\lambda}\cap S_\mu^-, \mathbb{C})$

$\Rightarrow$ Duistermaat–Heckman measure of $\overline{Z} = \iota_*(t_\lambda)_*\left(\sum_{\underline{i}} \langle v_\lambda^*, e_{\underline{i}}\, v_Z\rangle\, D_{\underline{i}}\right)$

translation by λ

linear form on $L(\lambda)$: 1 on hw vector
0 on other weight spaces

Ingredients: • action of e on v_Z (Ginzburg, Vasserot) given as $c_1(\mathcal{L})\cup -$ on $IH(\overline{\mathcal{G}r^\lambda}, \mathbb{C})$

where $\mathcal{L} = \mathcal{O}(1)|_{\mathcal{G}r}$ (depends on q)

• formula by Knutson relating DH measures and chains of Bialynicki-Birula cells.

3) Compute the Fourier-Laplace transform $\mathcal{M} \to \mathcal{H}ol(\underline{t}^*_{\mathbb{C}})$

$$FT(D_{\underline{i}}) = \sum_{p=0}^{n} \frac{e^{-\beta_p}}{\prod_{q\neq p}(\beta_q - \beta_p)} \qquad \text{where } \beta_p = \alpha^\vee_{i_1} + \dots + \alpha^\vee_{i_p} \quad (\text{depends on } \underline{i})$$

Set $\overline{D}_{\underline{i}} = \prod_{q=0}^{n-1} \frac{1}{\beta_q - \beta_n}$ coefficient of $e^{-\beta_n}$ in $FT(D_{\underline{i}})$

Define $\overline{D}: \mathbb{C}[N^\vee] \to \mathbb{C}(\underline{t}^*_{\mathbb{C}})$, $f \mapsto \sum_{\underline{i}} f(e_{\underline{i}})\, \overline{D}_{\underline{i}}$ (well-defined since $\underline{n}^\vee$ acts locally nilpotently on $\mathbb{C}[N^\vee]$)

Thus: if $Z \in \mathrm{Irr}(\overline{Gr^\lambda} \cap S^-_\mu)$ then $e^T_{L_\mu}(Z) = \iota^*(\overline{D}(\langle v^*_\lambda, -v_Z\rangle))$.

(Fact: $\overline{D}$ is an homomorphism of algebras, so corresponds to a rational map $\underline{t}^*_{\mathbb{C}} \dashrightarrow N^\vee$.
We shall characterize the image of a general point $x \in \underline{t}^*_{\mathbb{C}}$ by the following device.)

For $x \in (\underline{t}^\vee_{\mathbb{C}})^{reg}$: $N^\vee$ acts simply transitively on $x + \underline{n}^\vee$.
define $n_x \in N^\vee$ so that $Ad_{n_x}(x) = x + e$.

Prop: 1) $\overline{D}: \mathbb{C}[N^\vee] \to \mathbb{C}(\underline{t}^*_{\mathbb{C}})$ is the algebra map induced by $(\underline{t}^\vee)^{reg} \to N^\vee$, $x \mapsto n_x$.
$\underline{t}^*_{\mathbb{C}} = \underline{t}^\vee_{\mathbb{C}}$

2) For each $(x, w) \in (\underline{t}^\vee_{\mathbb{C}})^{reg} \times W$, the Gaussian decomposition of $n_x \dot{w} \in G^\vee$ has the form $y\, n_{w^{-1}x}\, t$ with $(y, t) \in (N^-)^\vee \times T^\vee$.
any lift of w in $N_{G^\vee}(T^\vee)$.

Note added after the talk in answer to a question from the audience

Assume V is a vector space with a linear action of a torus T, choose a basis $(v_0, .., v_n)$ of V consisting of weight vectors, denote by $\chi_i \in X^*(T)$ the weight of v_i. Over $\mathbb{C}$ the action of T on $\mathbb{P}(V)$ is hamiltonian with moment map $(x_0 : \cdots : x_n) \mapsto -\left(\sum_{i=0}^{n} |x_i|^2 \chi_i\right) \Big/ \left(\sum_{i=0}^{n} |x_i|^2\right)$.

Let $X \subset \mathbb{P}(V)$ be closed and T-invariant. For each $n \in \mathbb{N}$ write the class of the T-module $\Gamma(X, \mathcal{O}(n))$ in the representation ring of T as a formal sum $\sum_{\nu \in X^*(T)} d_\nu(n)\, e^\nu$, with $d_\nu(n) \in \mathbb{Z}$. (Alternatively, apply the push-forward to a point on the class $[\mathcal{O}_X \otimes \mathcal{O}(n)]$ in equivariant K-theory; this gives the same result for n large enough.) Then Brion and Procesi define the DH-measure of X as the weak limit $\lim_{n \to \infty} \frac{1}{n^{\dim X}} \sum_{\nu \in X^*(T)} d_\nu(n)\, \delta_{\nu/n}$, where $\delta_{\nu/n}$ is the Dirac mass at $\nu/n \in \underline{t}_{\mathbb{C}}^*$.

These constructions can be applied to an MV cycle $\overline{Z} \subset Gr$. Namely the quadratic form q defines a central extension $\widehat{G(\mathcal{K})}$ of $G(\mathcal{K})$, a Kac-Moody group with maximal parahoric $\widehat{G(\mathcal{O})}$, and the action of $\widehat{G(\mathcal{K})}$ on the highest weight line of the basic representation $V = L(\Lambda_0)$ provides the Borel-Weil-Bott embedding $\Psi : Gr \to \mathbb{P}(V)$. It maps the T-fixed point $L_\nu \in Gr$ to a line of V on which T acts by the character $-\iota(\nu)$; the moment map thus sends L_ν to $\iota(\nu)$. Note that this contributes to a term $e^{n\nu}$ in the class of $\Gamma(X, \mathcal{O}(n))$ if $L_\nu \in X$, hence to a Dirac mass δ_ν in the sum above.