DAHA superpolynomials etc.

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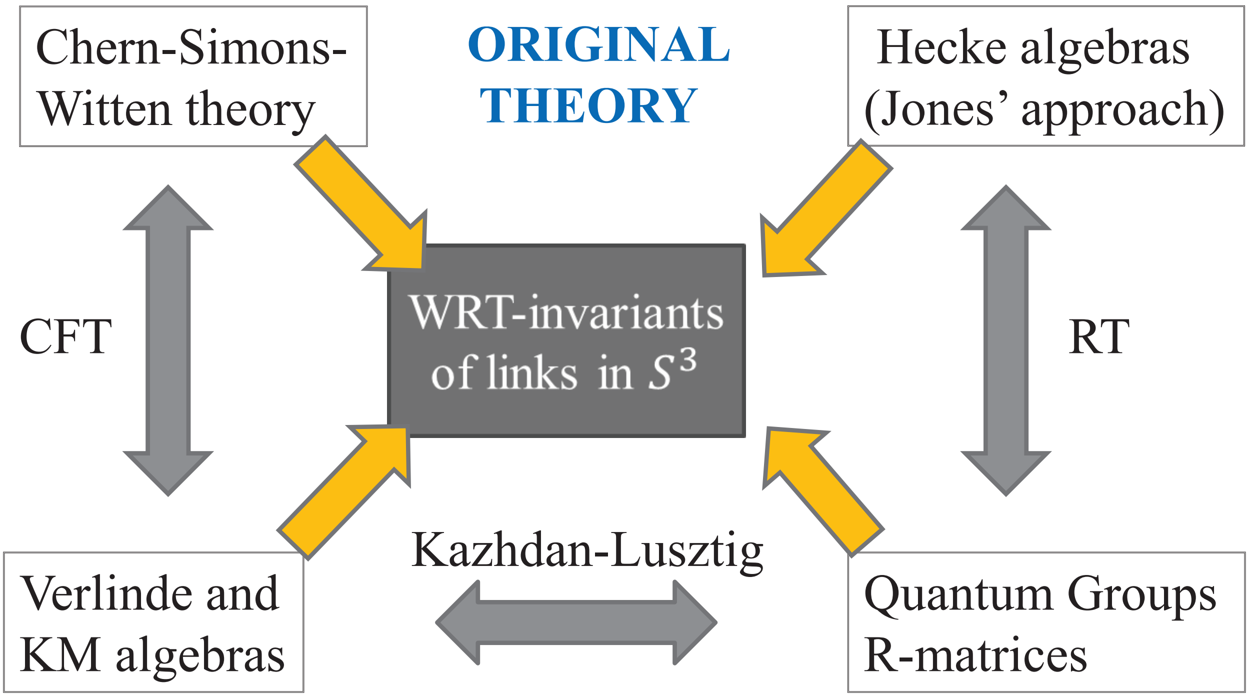
ABSTRACT. The DAHA superpolynomials are invariants of colored iterated torus links, generalizing the HOMFLY-PT polynomials. Presumably they can be defined for all 5 Deligne-Vogel series, but this is fully done only for type A beyond some particular knots. In the uncolored case and for iterated knots, they are conjectured to coincide with the stable reduced Khovanov-Rozansky polynomials, the most powerful numerical knot invariants we have. For uncolored torus knots, this is due to Elias, Hogancamp and Mellit. Also, DAHA superpolynomials conjecturally coincide with the motivic superpolynomials of plane curve singularities and satisfy certain Riemann Hypothesis.

Motivic superpolynomials are defined in terms of compactified Jacobians of plane singularities. They are conjectured to coincide with (flagged) *L*-functions of plane curve singularities, the numerators of the ζ-functions due to Galkin and Stöhr. Considering the equations of plane singularities as superpotentials in *Landau-Ginzburg theory*, **the motivic superpolynomials are expected to give the corresponding partition functions.** *The functional equation for the zeta-functions (proven), conjecturally coinciding with the super-duality of the DAHA superpolynomials (proven too), appear therefore connected with the S-duality in M-theory,* **which seems a fundamental connection between physics, geometry and number theory. Motivic superpolynomials satisfy Riemann Hypothesis for sufficiently small q, which can be hopefully (??) related to "phase transitions" in LGSM, at least by analogy with the Lee-Yang theorem from spin chains.**

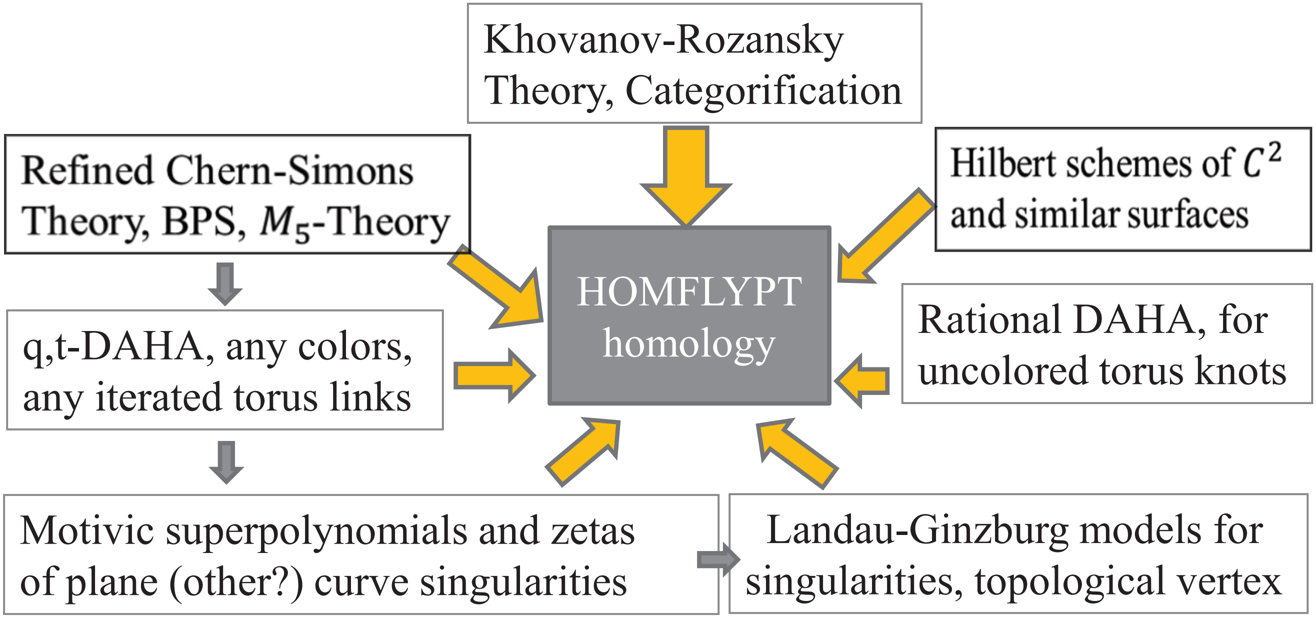
KNOT INVARIANTS



Stabilization at roots of unity q remains quite a challenge! Needed for the invariants of 3-folds (lens, Seifert spaces).

NEW VINTAGE

REFINED THEORY



Big Puzzle: Unification of these and related approaches:

Combinatorics, Affine Springer Fibers, Instanton sums, Spin chains, … .

TOWARD "MOTIVIC LGSM"

(1) The super-duality of the *physics superpolynomials* is related to the S-duality via M-theory. DAHA provide general algebraic framing for Fourier theory and deformed/nonsymmetric/ ... Verlinde algebras. DAHA superpolynomials are clearly correlation functions.
(2) Vafa-Warner's paper "Catastrophes..." (1989) promoted study of Landau-Ginzburg Sigma Models as directly as possible in terms of the *singularities* of superpotentials $W(x,y)$ (can be with 3-4 variables). E.g. $\mu = (r-1)(s-1)$ is the number of chiral operators (=Milnor number=Witten index), $c=6\big((\frac{1}{2}-\frac{1}{r})(\frac{1}{2}-\frac{1}{s})\big)$ for $W=x^r-y^s$, etc.
(3) The relation between $LGSM$ and $SCFT$ suggests that the S-duality can be seen via W. The (conjectural) coincidence of the DAHA superpolynomials and MOTIVIC ones implies that physics S becomes the functional equation for (at least) the singularities $W(x,y)=0$ at $x=0=y$ considered over $\mathbb{F}_q$, with t being essentially T from the zeta and the corresponding L-function (its numerator).

REFINED KNOT OPERATORS

1. The origin was [*M.Aganagic, S.Shakirov, 2011*]. They replaced Schur functions in the knot operators by Macdonald polynomials at roots of unity (for $q^m = 1; t = q^k, k \in \mathbb{Z}_+$ in the DAHA parameters), and conjectured: (a) m-stabilization, (b) stabilization with respect to SL_N, and (c) coincidence with the KhR polynomials. The q-stabilization is generally a difficult task, but the main problem **was that the** *refined Verlinde S-operator* **(Cherednik-Kirillov) requires the usage of ALL Macdonald polynomials ($\sim m^{N-1}$, to be more exact), which is not realistic theoretically and practically beyond very small m, N. And the S-operator alone is insufficient here.**

2. This was fixed in [*I.Ch, 2013*]. Refined WRT polynomials were defined (for any root systems). The coincidence of the DAHA polynomials at $q=t$ for torus knots with the HOMFLY-PT ones was proved there via CFT (*S.Stevan*, etc.) and Hecke algebras. The same proof actually works for the Kauffman polynomials.

TOWARD HOMFLY-PT HOMOLOGY

HOMFLY-PT POLYNOMIALS = DAHA ONES AT $\mathbf{t} = \mathbf{q}$.

3. [*I.Ch., I.Danilenko, 2014*] **contains a complete proof of "=" for Jones** (A_1) **polynomials for torus iterated** *knots* **based on the Rosso-Jones formula (with exact framing factors). This can be extended to any colored HOMFLY-PT polynomials** (*I.D, unpublished*).

4. [*H.Morton, P.Samuelson, 2015*]. **Any torus iterated** *knots*; **based on the identification of the** skein of the torus **with the** elliptic Hall algebra **(due to** *Burban-Schiffmann-Vasserot*) **at** $q = t$.

5. It was then extended to iterated torus *links* **in** [*I.Ch, I.D, 2015*] **using the "Seifert framing", generalizing that from** [*MS*].

KHR POLYNOMIALS = DAHA ONES (ANY q, t).

Soergel bimodules and Gorsky's (combinatorial) formulas were used:

6. This identification was done in [*B.Elias, M.Hogancamp, 2016-17*] **for** $T(mr \pm 1, r), T(mr, r)$ **and in** [*A.Mellit, 2017*] **for any torus knots.** *Iterations, links and colors remain quite a challenge.*

REFINED HOMFLY-PT POLYNOMIALS

Algebraic links will be mostly considered. They are intersections of *plane curve singularities* $0 \in \mathcal{C} \subset \mathbb{C}^2$ **with small** $\mathbb{S}^3$ **centered at** 0**. They have a natural orientation from that of** $\mathcal{C}$**. For instance, torus knots/links** $T(r,s)$ **are for the singularities** $x^r - y^s = 0$ $(r,s>0)$**. For any links, the (uncolored)** *HOMFLY-PT polynomials* $H(q,a)$ **are defined in the** *DAHA parameters* **as follows:**

$$a^{1/2} H(\nwarrow\!\!\!\!\nearrow) - a^{-1/2} H(\nearrow\!\!\!\!\nwarrow) = (q^{1/2} - q^{-1/2}) H(\uparrow\uparrow), \;\; H(\bigcirc) = 1.$$

Their t**-refinements are the DAHA-superpolynomials: a theorem for** *colored* **torus iterated links. Conjecturally they coincide with the corresponding** *Khovanov-Rozansky stable reduced polynomials***. The theory of these polynomials and HOMFLY-PT homology mostly exists by now for uncolored** *knots* **and in the unreduced setting; very few formulas for them are known beyond (uncolored)** *torus* **knots.**

HOMFLY-PT INVARIANTS via DAHA

Generally, tilde-normalization is $\widetilde{H} = H^{\sim} \stackrel{\text{def}}{=\!=} q^{\bullet}t^{\bullet}a^{\bullet}H \in 1 + q\mathbb{Z}[[q]] + a\mathbb{Z}[[q^{\pm1}, t^{\pm1}, a]]$**. For HOMFLY-PT** $H(q,a)$ **and** *uncolored* **links with** κ **components, let** $\widetilde{H} = (1-q)^{\kappa-1}H(q,a)^{\sim}$**.**

THM. For DAHA super-polynomials $\mathcal{H}$ **of arbitrary torus iterated** *colored* **links** $\widetilde{H}(a,q) = \mathcal{H}(t=q, a \mapsto -a)$**.**

	HOMFLY-PT $H(q,a)$	$\widetilde{H}$ **("tilde**$-H$**")**	**DAHA super-**$\mathcal{H}$
$T(3,2)$:	$a(q+q^{-1}-a)$	$1+q^2-qa$	$1+qt+qa$
$\mathscr{X}$:	$a^{1/2}\,\frac{1+a-q-q^{-1}}{q^{1/2}-q^{-1/2}}$	$1-q+q^2-qa$	$1+qa+(q-1)t$,

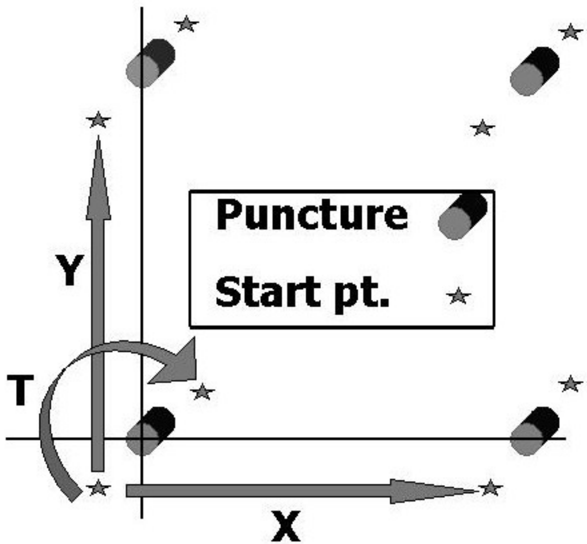
where $\mathscr{X}$ **is Hopf-plus-link. Also, tilde-**$WRT_{SL_N} = \widetilde{H}(q, a=q^N)$ **(Jones for** $N=2$**), Alexander**$^{\sim} = \widetilde{H}(q, a=1)/($ **extra** $(1-q)$ **if** $\kappa>1$**). Jones**$^{\sim}(T(3,2), \mathscr{X}) = 1+q^2-q^3, 1+q^2$**; Alexander**$^{\sim}(T(3,2), \mathscr{X}) = 1-q+q^2, 1$**. The latter 2 are direct from the plane curve singularities.**

ELLIPTIC CONFIGURATION SPACE

For $E = T^2$, we set $\mathcal{H} = \mathbb{C}\mathbf{B}_{ell}/\{T_i^2 + aT_i + b = 0\}$ for $\mathbf{B}_{ell} = \pi_1\big((E^N \setminus \{x_i = x_j\})/\mathbf{S}_N\big)$; $T_i (1 \le i < N)$ are the usual "half-turns". $\mathcal{H}$ can be generalized to any root systems, but then orbifold π_1 must be used. Generally, "Non-commutative Kodaira-Spencer": for a manifold X, $\pi_1(\mathcal{M}_X)$ acts in (individual) $\pi_1(X)$ by outer automorphisms modulo inner (instead of using $H^1(X, \mathcal{T}X)$). Here *projective* **$PSL_2(\mathbb{Z})$ ($= B_3$ due to Steinberg) acts in $\mathcal{H}$, which is far from obvious in other approaches: via $K_{T\times C^*}(\widehat{G/B})$ and Harmonic Analysis.**

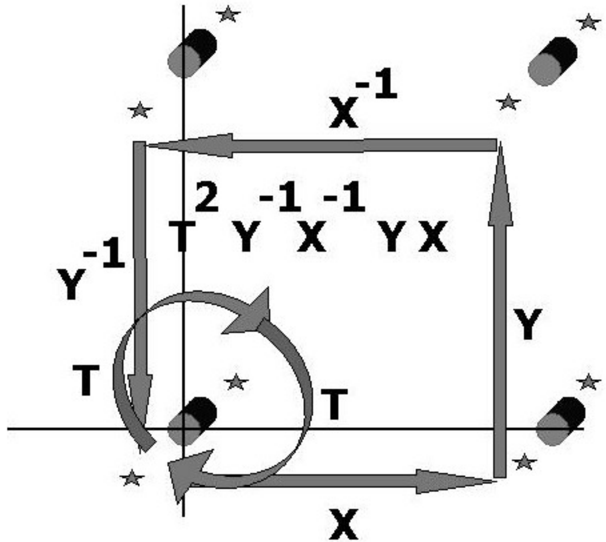
DAHA is a universal flat deformation of the Heisenberg-Weyl algebra extended by W; its Fock representation is the *polynomial representation*. **For curves C of genus > 1, the corresponding $\mathbf{B}_C$ (Birman, Scott) are "non-integrable": do not have Fock representations.**

ORBIFOLD π_1 FOR A_1



$$\mathbf{B}_{ell} = \langle X, Y, T\rangle / \{TXTX^{-1}, TY^{-1}TY, T^2Y^{-1}X^{-1}YX\}.$$

RELATION $T^2Y^{-1}X^{-1}YX = 1$



No punctures inside the path!

$$\textbf{A}_1\textbf{-DAHA: } \mathcal{H} \stackrel{\textbf{def}}{=\!=} \langle T, X^{\pm 1}, Y^{\pm 1}, t^{\pm\frac{1}{2}}, q^{\pm\frac{1}{4}} \rangle$$

subject to relations: $TXTX = 1 = TY^{-1}TY^{-1}$,

$Y^{-1}X^{-1}YXT^2 = q^{-1/2}, \ (T - t^{\frac{1}{2}})(T + t^{-\frac{1}{2}}) = 0;$

$\widetilde{PSL_2(\mathbb{Z})} \ni \tau_\pm, \ \tau_+ : Y \mapsto q^{-\frac{1}{4}} XY, X \mapsto X, T \mapsto T.$

For $t=1$, $\mathcal{H}$=Weyl algebra $\rtimes \mathbf{S}_2$ under $T \to s$.

$$\mathcal{H} \circlearrowright \mathbb{C}[X^{\pm 1}] : \ \ T \mapsto t^{1/2} s + \frac{t^{1/2} - t^{-1/2}}{X^2 - 1}(s-1),$$

$X \mapsto X, \ \ Y \mapsto spT, \ \ s(X) = X^{-1}, \ p(X) = q^{1/2}X,$

For GL_n, $\tau_+(Y_1) = q^{-1/2} X_1 Y_1$, $\tau_-(X_1) = q^{+1/2} Y_1 X_1$,
$Y_1 = \pi T_{n-1} \ldots T_1$, $\pi : X_1 \mapsto X_2, \ldots, X_n \mapsto q^{-1} X_1, \ldots$

DAHA via HARMONIC ANALYSIS

THM: $L=\sum_{0\le i\le N}\partial_i^2+k\sum_{1\le i<j\le N}V(x_i-x_j)$ **has ∞-many conservation laws (similar M such that $[L,M]=0$) iff** $(a)\,V=\frac{1}{(x_i-x_j)^2},\ (b)\,V=\frac{1}{\sinh^2(x_i-x_j)},\ (c)\,V=\wp(x_i-x_j):$ **rational & rational(a), rat & hyperbolic(b), rat & elliptic(c) theories.**

Self-dual hyperbolic &hyperbolic Macdonald theory:

$M_m=\sum_I\prod_{i\in I\ j\notin I}\frac{t^{1/2}q^{x_i}-t^{-1/2}q^{x_j}}{q^{x_i}-q^{x_j}}\Gamma_{i_1}\cdots\Gamma_{i_m}$ **all commute, where** $I=(i_1,\ldots,i_m),1\le i_1<\cdots<i_m\le N$**,** $\Gamma_i(x_j)=\delta_{ij}+x_j$**.**

DAHA simplify and integrate the eigenvalue problem for $\{M_m\}$. Furthermore, the trigonometric&elliptic theory was obtained. The "ultimate" ell&ell theory would be parallel to the $6d,N=2$ theory (X-theory). The LP is for $d=4$; it corresponds to the Whittaker limit $t=0$ of Macdonald theory.

REFINED VERLINDE ALGEBRAS: $q=\exp(\frac{2\pi i}{N})$, $k<N/2, k\in\frac{\mathbb{Z}_+}{2}$. **The map** $X(z)=q^z$ **can be extended to an** $\mathcal{H}$**-homomorphism** $\mathbb{C}[X^{\pm 1}]\to V \stackrel{\textbf{def}}{=\!=}$ **"Nonsym Verlinde"=** $Funct\{-\frac{N+k+1}{2},\ldots,-\frac{k+1}{2},-\frac{k}{2},\frac{k+1}{2},\ldots,\frac{N-k}{2}\}$.

Moreover, X,Y,T **are unitary in** V**, which requires the "minimal" primitive** N**th root** q**. Also,** $PSL_2(\mathbb{Z})$ **acts in** V **projectively and in the image** $V_{sym}=\{f\mid Tf=t^{\frac{1}{2}}f\}$ **of** $\mathbb{C}[X^{\pm 1}]_{sym}$**. So** $\dim_{\mathbb{C}}V=2N-4k$, $\dim_{\mathbb{C}}V_{sym}=N-2k+1$. **Usual** *Verlinde algebra* **is** $V_{sym}^{k=1}$**;** τ_+ **becomes the** T**-operator,** $\sigma=\tau_+\tau_-^{-1}\tau_+$ **, Fourier transform , becomes the** S**-operator.** *"Characters"* **in Verlinde algebras are replaced by eigenfunctions of** Y **and** $Y+Y^{-1}$ **in** V **and** V_{sym}**, the images of the Macdonald polynomials. Connections with minimal models of Kac-Moody algebras and** W**-algebras are expected.**

$\widetilde{\mathrm{PSL}(2,\mathbb{Z})} \circlearrowright \mathcal{H\!H} \Rightarrow$ invariants of $T(r,s)$

Torus knot $T(r,s)$, $r,s>1$, $(r,s)=1$, $r,s=$**winds** **(horizontal, vertical), or** $T=\{x^r=y^s\}\cap S^3_\epsilon\subset\mathbb{C}^2$.

$$T(r,s) \leftrightsquigarrow \gamma = \gamma_{r,s} = \begin{pmatrix} r & * \\ s & * \end{pmatrix} \in SL_2(\mathbb{Z}).$$

`Construction (Aganagic-Shakirov, Cherednik):`

1. **DAHA** $\mathcal{H\!H}_{q,t}$ **for a root system** $R\subset\mathbb{R}^n$,
2. **a dominant weight** $\lambda\in P_+=\oplus_{i=1}^n\mathbb{Z}_+\omega_i$,
3. **and the Macdonald polynomial** $P_\lambda(X)$,
4. **Coinvariant:** $H\in\mathcal{H\!H}$, $\{H\}\stackrel{\mathbf{def}}{=\!=}H(1)(X\mapsto t^{-\rho})$.
5. $DJ(\lambda;q,t_\bullet)=$ **Coinvariant** $\left(\widetilde{\gamma}(P_\lambda/P_\lambda(t^\rho))\right)$,
6. $\mathcal{H}^{DAHA}_{T(r,s)}(\lambda,q,t,a=-t^{n+1})=\widetilde{DJ}(A_n;\lambda;q,t)$.

$$\widetilde{DJ}^{A_n}_{T(3,2)}(\square)=\{\tau_+\tau_-^2(X_1)\},\ \mathcal{H}^{DAHA}_{T(3,2)}(\square)=1+aq+qt.$$

USING COINVARIANT $(A_1, \rho = 1/2)$

Coinvariant: $H \in \mathcal{HH}$, $\{H\} \stackrel{\mathbf{def}}{=\!=} H(1)(X \mapsto t^{-\frac{1}{2}})$.

$$DJ_{3,2} = \{\tau_+\tau_-^2(X)\} \sim \{(XY)(XY)X(1)\} \sim \{Y(X^2)\}$$

$$= t^{-\frac{1}{2}} q^{-1} X^2 - t^{\frac{1}{2}} + t^{-\frac{1}{2}}|_{X^2 \mapsto t^{-1}} \sim 1 - qt^2 + qt;$$

Jones Polynomial of $T(3,2)$: $\widetilde{DJ}_{3,2}(t \mapsto q) = 1 + q^2 - q^3$.

We use here $E_1 = X$ **instead of** $P_1(X)$**:** $Y(X) = (qt)^{-\frac{1}{2}} X$. **Using E-polynomials is an important feature of the theory! By $\sim$ we mean "up to $q^\bullet t^\bullet$". Extending this to A_n and super-polynomials:** $\mathcal{H}_{3,2}^{\square} = 1 + aq + qt$; **e.g.** $\mathcal{H}_{3,2}(a \mapsto -t^2) = \widetilde{DJ}_{3,2}$.

This is equally simple for $T(2m+1, 2)$ **with** $m > 0$**; for such torus knots** $deg_a = 1$. **Generally,** $deg_a \mathcal{H}_{r,s}^{\lambda} = |\lambda|(\mathbf{Min}(r,s) - 1)$.

SUPERPOLYNOMIALS FOR T(2p+1,2):

$$\mathcal{H}_{2p+1,2}(m\omega_1) = \frac{(q;q)_m}{(-a;q)_m(1-t)} \sum_{k=0}^{m} (-1)^{m-k}$$

$$(qt)^{\frac{m-k}{2}} \Big((q^{\frac{m(m+1)}{2}} - q^{\frac{k(k+1)}{2}})(t/q)^{\frac{m-k}{2}} \Big)^{2p+1}$$

$$\frac{(t;q)_k(-a;q)_{m+k}(-a/t;q)_{m-k}(1-q^{2k}t)}{(q;q)_k(qt;q)_{m+k}(q;q)_{m-k}};$$

$$\mathcal{H}_{3,2}(m\omega_1) = \sum_{k=0}^{m} q^{mk} t^k \frac{(q;q)_m(-a/t;q)_k}{(q;q)_k(q;q)_{m-k}};$$

$$(a;q)_n = (1-a)\cdots(1-aq^{n-1}).$$

Proposed by [Dunin-Barkowski- Mironov- Morozov- Sleptsov- Smirnov, 2011-12], [Fuji- Gukov- Sulkowsky, 2012]. Habiro's formula (2000) is for $p=1, a=-t^2, t=q$**. Proved via DAHA.**

ITERATIONS. **E.g., let the singularity ring be** $\mathcal{R}=\mathbb{C}[[x=z^8, y=z^{12}+z^{14}+z^{15}]]$. **Then arith.genus**$(\delta)=42$**, valuation semigroup** $\Gamma=$ **val**$_z(\mathcal{R})=\langle 8,12,26,53\rangle$. **Newton's pairs are:** $\{(3,2),(2,1),(2,1)\}$. **The corresponding singularity is** $\mathcal{C}\simeq\{x=y^{\frac{2}{3}}(1+c_1 y^{\frac{1}{3\cdot 2}}(1+c_2 y^{\frac{1}{3\cdot 2\cdot 2}}))\}$**, the link** $\mathcal{C}\cap S^3_\epsilon$ **is** $\mathcal{L}=Cab(53,2)Cab(13,2)T(3,2)$**, and:**

$$DJ^\lambda_{\mathcal{L}}=\{\mathcal{P}_\lambda\},\mathcal{P}_\lambda=\Downarrow\begin{pmatrix}3 & *\\ 2 & *\end{pmatrix}\Downarrow\begin{pmatrix}2 & *\\ 1 & *\end{pmatrix}\Downarrow\begin{pmatrix}2 & *\\ 1 & *\end{pmatrix}\Big(\frac{P_\lambda(X)}{P_\lambda(t^{-\rho})}\Big),$$

where the matrices act via their lifts to Aut$(\mathcal{H\!H})$**, and** $\Downarrow H\overset{\textbf{def}}{=\!=}H(1)$, $\{H\}\overset{\textbf{def}}{=\!=}H(1)(t^{-\rho})$ **is the coinvariant for** $1\in\mathbb{C}[X]$, $P_\lambda=$ **Macdonald polynomial for a dominant weight** λ **(a partition for superpolynomials in type** A**). Upon the** a**-stabilization, deg**$_a(\mathcal{H}^\square)=Min(3,2)\cdot 2\cdot 2-1=7=$ **multiplicity**$(\mathcal{C})-1$.

ROSSO-JONES RELATIONS

For any $m\in\mathbb{Z}$**,** $r,s\ge 0$**, let** $\mathcal{H}^{\square}_{[m;r,s]}$ **be the uncolored DAHA superpolynomial for the cable** $Cab(2rs{+}2m{+}1,2)T(r,s)$**, under the** *tilde-normalization***:** $\mathcal{H}(a{=}0)=1{+}q(\cdot){+}t(\cdot)$**. Then the** q,t**-Rosso-Jones formula reads:** $\mathcal{H}^{\square}_{[m;r,s]}=$

$$=\frac{1{+}aq}{1{-}qt}\left(1{+}(qt)^m\frac{q(1{-}t)}{1-q}\right)\mathcal{H}^{\square\square}_{r,s}-(qt)^m\frac{q}{1{-}q}\mathcal{H}^{\square}_{2r,2s}$$

$$=\frac{1{-}(qt)^m}{1{-}qt}(1{+}aq)\mathcal{H}^{\square\square}_{r,s}+(qt)^m\mathcal{H}^{\square}_{[0;r,s]}\quad \textbf{(a theorem)},$$

where $\square\square$ **means "colored by** $2\omega_1$**"; positive for algebraic** $(m\ge 0)$**. Say, for (non-algebraic)** $K=Cab(3,2)T(3,2):\ \ \mathcal{H}^{\square}_K=\ 1-q^2+qt+q^2t-q^3t+q^2t^2+q^3t^3+a^3\left(-\frac{q^4}{t^2}-\frac{q^5}{t}\right)+a^2(q^3-q^4-q^5-\frac{q^3}{t^2}-\frac{q^3}{t}-\frac{2q^4}{t})+a\left(q+q^2-2q^3-q^4-\frac{q^2}{t}-\frac{q^3}{t}+q^2t+q^3t-q^4t+q^3t^2\right).$

TORUS ITERATED LINKS

An iterated link is represented by a tree $\mathcal{T}$ or a pair of trees $\{\mathcal{T},\mathcal{T}'\}$ (or their unions) with fixed origin and the vertices labeled by relatively prime $[r,s]$. This is based on splice diagrams (Neumann, ...), where colors (Young diagrams) are assigned to the *arrows* **added at the ends of the** *paths* **in $\mathcal{T},\mathcal{T}'$. When two paths for λ,μ meet at $\circ_{[r,s]}$, we employ $\widetilde{\gamma}=\widetilde{\gamma}_{r,s}$ to the product $\mathcal{P}_\lambda\mathcal{P}_\mu$ (for $\widetilde{\gamma}_{r,s},\mathcal{P}_\lambda$ as above) and then continue constructing** *pre-polynomials* **$\mathcal{P}$ by induction. Finally, $\mathcal{H}_{DAHA}$ is $\{\mathcal{P}\}$ or $\{\mathcal{Q}(Y^{\pm1})\mathcal{P}\}$ for $\mathcal{T},\mathcal{T}'$ with the pre-polynomials $\mathcal{P},\mathcal{Q}$; this is upon the a-stabilization. Here Y^{+1} must taken for algebraic links (we omit the other conditions for algebraic links). Pre-polynomials themselves are invariants of links in a solid torus.**

Ex. Let $\mathcal{Q}=P_\mu$ (i.e. $\mathcal{T}'$ is the μ-arrow with *no vertices***). Applying $P_\mu(Y^{\pm1})$ corresponds to adding a** *meridian* **colored by μ to the link for $\mathcal{T}$ (±1 gives its orientation); $\circ_{[1,\pm1]}\overset{\mu}{\underset{\mathcal{T}}{\rightrightarrows}}$ can be used here too.**

3 EXAMPLES OF LINKS

1) The cable $\big(Cab(11,3), Cab(11,3)\big)T(3,1)$ **is represented by the tree** $\circ \rightrightarrows \substack{\circ \\ \circ} \rightrightarrows \lambda$**, where the first vertex is labeled by** $[3,1]$ **and the other two by (coinciding) labels** $[3,2]$**. The superpolynomial is then** $\mathcal{H} = \Big\{\widetilde{\gamma}_{3,1}\Big(\big(\widetilde{\gamma}_{3,2}(P_{\lambda}^{\maltese})\big)^{2}\Big)\Big\}$ **for** $(x^{11}-y^3)((y+x^3)^3+x^{11}) = 0$ **with the linking number** 27 $[I.Ch, "RH"]$**. The normalization** $\maltese$ **in** $\mathcal{H}_{DAHA}$ **is non-topological: to ensure the polynomiality we use** *proper* ***J*-polynomials** **instead of** P_λ **and divide by the** LCM **of all** $J_\lambda(t^{-\rho})$**.**

2) Adding the meridian(+) to $T(3,2)$ **gives** $\mathcal{H} = \{P_{\omega_1}(Y)\widetilde{\gamma}_{3,2}(P_{\omega_1})\} \sim 1-t+qt+q^2t-qt^2+q^3t^2-q^2t^3+q^3t^3-q^3t^4+q^4t^4+a^2(q^3-q^3t+q^4t)+a(q+q^2-qt+2q^3t-q^2t^2+q^4t^2-q^3t^3+q^4t^3)$ **for** $(y^3+x^2)(y^3+x)=0$**.**

3) The associativity&invariance of the $\mathcal{H}$ **for Hopf 3-links, theory of DAHA-vertex, results in a** *refined* $TQFT$**. E.g. for** $T(3,-3)$ **(with linking numbers** -1**), they are** $\{\widetilde{\gamma}_{1,-1}(P_\lambda P_\mu P_\nu)\}$ $[I.Ch, I.D]$**.**

RIEMANN HYPOTHESIS

We substitute $q \to qt$: $\mathbf{H}(q,t,a) \stackrel{\mathbf{def}}{=\!=} \mathcal{H}_{DAHA}(qt,t,a)$, $\mathbf{H}(q,t,a) = \sum_{i=0}^{deg_a} \mathbf{H}_i(q,t)a^i$. Then $\mathbf{H}(q \to q, t \to 1/(qt), a) = q^{\cdot} t^{\cdot} \mathbf{H}(q,t,a)$, which is DAHA super-duality [Ch, Gorsky, Negut, Ch-Danilenko]; thus if $t = \xi$ is a zero of $\mathbf{H}_i$, then so are $1/(q\xi)$ and (obviously) $\overline{\xi}$.

RH: For *uncolored* algebraic knots, all t-zeros ξ of $\mathbf{H}_i(q,t)$ satisfy $|\xi| = \sqrt{1/q}$ for $0 \le q \le \kappa$; where $\boldsymbol{\kappa = 1/2}$ is sufficient for $\boldsymbol{i = 0}$ ("quantitative RH"). For algebraic *links* with p components, $p-1$ *non-RH* pairs are conjectured for $\mathbf{H}_{i=0}$; but $\mathbf{H}_{i=2}$ has 3 such pairs for $\{(y^3 - x^2)(x^3 - y^2) = 0\}$. For torus knots, all $\mathbf{H}_i(q=1,t)$ are products of cyclotomic polynomials (related to Shuffle Conjecture).

The techniques of $[I.Ch, "RH"]$ allow to calculate the number of non-RH zeros of $\mathbf{H}_i$ for any colored iterated links (conjecturally, there is none for rectangle diagrams). The existence of κ can be proven for uncolored *motivic* superpolynomials ("qualitative RH").

REFINED WITTEN INDEX

The KhR polynomials for SL_N are related to $\mathcal{H}(q,t,a=-t^N)$, but this requires the differentials d_N (Khovanov-Rasmussen), which are generally involved. For $a=-1$, this connection is with knot Floer homology; $\mathcal{H}(t,t,a=-1)$ is the Alexander polynomial. For algebraic knots, $\delta_{q,t} \overset{\textbf{def}}{=\!=} \frac{\mathcal{H}(q,t,a=-t/q)-(qt)^{\delta}}{1-t}$ is a q,t-deformation of the $\frac{\text{Milnor number}}{2} = \frac{\text{Witten index}}{2}$; δ is the arithmetic genus of the corresponding plane curve singularity. Representing it by $\mathcal{R} \subset \mathbb{C}[\![z]\!]$, let $\Gamma \overset{\textbf{def}}{=\!=} val_z(\mathcal{R})$, $\mathbb{Z}_+ \setminus \Gamma = \cup_{i=1}^{\nu}[g_i, g_i']$, a union of segments of length $m_i = g_i' - g_i + 1$. Then $\delta = \sum_{i=1}^{\nu} m_i$, $\delta_{q,t} = \frac{1-t^{g_1}}{1-t} + \sum_{i=1}^{\nu-1} \frac{t^{g_i'+1}-t^{g_i+1}}{1-t}(\frac{q}{t})^{m_1+\cdots+m_i}$ (for Gorenstein $\mathcal{R}$). By analogy with spin-chains, the corresponding LGSM can be expected "stable" at q if RH (as above) holds (Lee-Yang thm), but this is a long shot.

LEE-YANG THEOREM

For any lattice (any d) with the connected pairs denoted by $\langle n, n' \rangle$ and the number of vertices N, let $LIM = \lim_{N\to\infty} \frac{\log(Z_N)}{N}$ for $Z_N = \sum_{\{\sigma_n\}} e^{-\beta\mathcal{H}}$, where $\mathcal{H} = -\sum_{\langle n,n'\rangle} J_{n,n'}\sigma_n\sigma_{n'} - H\sum_n \sigma_n$ and $\sigma = \pm 1$ (Ising model with external magnetic field H). Here $\beta = (k_B T)^{-1}$ etc. Assuming that $J_{n,n'} \geq 0$ and $\beta > 0$, Lee and Yang proved that the zeros of Z in terms of $\mu = e^{-2\beta H}$ belong to the unit circle $|\mu| = 1$. For square lattice with $J = const$, the sum Z is a polynomial in terms of μ and $u = e^{-4\beta J}$. Experiments [*Matveev, Shrock, ...*] **show that $|\mu| = 1$ for the zeros holds even for $0 > u > u_{crit}$. This resembles the behavior of our t-zeros as q increases.**

Concerning LIM, the unimodularity of zeros (mostly) translates into the phase transition at $\mu = 1$ (and no other *real* ones!). When $u < u_{crit}$, the analytic properties of LIM in terms of $\mu \in \mathbb{R}_+$ become messy; "RH" for Z_N seems really to control of the physics stability.

PHYSICS STABILITY via RH?

Briefly: zeta-functions of target spaces in LGSM (say for local singularities) are expected important partition functions, which is not only for *conformal* **FT (here $\mathcal{N}=(2,2)$, but this can be potentially extended to $\mathcal{N}=(2,0)$). This can shed light on the** *modularity phenomenon* **in string theory; the** *functional equation* **is very universal in arithmetic geometry (though the uniformity w.r.t. q is very special!). In "Motivic LGSM" they are directly linked!**

Following the insight from *phase transitions*, **LGSM (and the corresponding elementary particles, the ground states) can be expected "stable" at $0 \le q \le \kappa$, a** *coupling constant*, **if RH holds (for t-zeros). Rationale: (1) small $q>0$ always satisfy RH ("*qualitative RH*"), (2) the simplest** *non*-**torus singularities are the most RH-unstable, i.e. have the smallest q violating RH, which is "*quantitative RH*".**

HILBERT SCHEMES AND JACOBIANS

$\mathcal{C}=$ unibranch plane curve singularity; $\delta=$arithmetic genus.

For rational $C\subset\mathbb{C}P^2$ (Gopakumar-Vafa, Pandharipande-Thomas): $\sum_{n\ge 0} q^{n+1-\delta} e(C^{[n]})=\sum_{0\le i\le\delta} n_C(i)\left(\frac{q}{(1-q)^2}\right)^{i+1-\delta}$**, for Euler numbers of** *Hilbert schemes* $C^{[n]}$; $n_C(i)\in\mathbb{Z}_+$ **(Göttsche, ..., Shende $\forall i$).**

ORS-Conjecture: For NESTED $\mathcal{C}^{[l\le l+m]}$ **Hilbert schemes (pairs of ideals) and $t\leftrightsquigarrow \mathfrak{w}=$***weight filtration* **(Serre, Deligne),**

$$\sum_{l,m\ge 0} q^{2l}\,a^{2m}\,t^{m^2}\,\mathfrak{w}(\mathcal{C}^{[l\le l+m]})\sim KhR^{\text{stab}}(Link(\mathcal{C})).$$

It adds t to Oblomkov-Shende conjecture, proved by Maulik.

ChD-Conjecture: For any $\mathcal{C}$, $\mathcal{H}_{DAHA}(\Box;a,q,t)=KhR^{\text{stab}}_{red}$, $\mathcal{H}_{DAHA}(\Box;q,t=1,a=0)=\sum_{i=0}^{\delta} q^i b_{2i}(\overline{J}(\mathcal{C}))$ (incl. $b_{2i+1}=0$).

$\overline{J}(\mathcal{C})$ = Jacobian factor, the key in Fundamental Lemma.

AN EXAMPLE **(b_{2i} via DAHA).** $\mathcal{R}=\mathbb{C}[[z^8, z^{12}+z^{14}+z^{15}]]$:

$$q^{-\delta}\mathcal{H}(\square; q, t=1, a=0) = 1+7q^{-1}+24q^{-2}+56q^{-3}+104q^{-4}+166q^{-5}$$
$$+\,236q^{-6} + 306q^{-7} + 370q^{-8} + 424q^{-9} + 465q^{-10} + 492q^{-11}$$
$$+\,507q^{-12} + 510q^{-13} + 504q^{-14} + 488q^{-15} + 466q^{-16} + 437q^{-17}$$
$$+\,406q^{-18} + 370q^{-19} + 335q^{-20} + 298q^{-21} + 264q^{-22} + 230q^{-23}$$
$$+\,199q^{-24} + 168q^{-25} + 143q^{-26} + 118q^{-27} + 97q^{-28} + 78q^{-29}$$
$$+\,63q^{-30} + 48q^{-31} + 38q^{-32} + 28q^{-33} + 21q^{-34} + 15q^{-35} + 11q^{-36}$$
$$+\,7q^{-37} + 5q^{-38} + 3q^{-39} + 2q^{-40} + q^{-41} + q^{-42}.$$

The Euler number of $\overline{J}(\mathcal{C})$ is 8512 ($q=1$), $\delta = 42$, and $\mathcal{H}(\square; q=p^l, t=1, a=0) = |\overline{J}(\mathcal{C})(\mathbb{F}_{p^l})|$, i.e. it coincides with the corresponding p-adic orbital integral. It doesn't depend on the matrix rank (15 or 8), only on the topological type of the (germ of the) *spectral curve* $\mathcal{C} \sim \mathcal{R}$**.**

MOTIVIC SUPERPOLYNOMIALS

Let $\mathcal{R}\subset\mathbb{C}[[z]]$ be the ring of a unibranch plane curve singularity. Flagged compactified Jacobian $\mathcal{F}$ is formed by *standard* **flags of $\mathcal{R}$-modules $M_0\subset M_1\subset\cdots\subset M_\ell\subset\mathcal{O}=\mathbb{C}[[z]]$ such that $(a)\,M_i\ni 1+z(\cdot)$, $(b)\,$dim$M_i/M_{i-1}=1$ and $M_i=M_{i-1}\oplus\mathbb{C}\,z^{g_i}(1+z(\cdot))$, $(c)\,(important)\,g_i<g_{i+1},\ i\geq 1$.**

Conjecture (Ch, Philipp). Within its topological type, the singularity can be assumed over any $\mathbb{F}=\mathbb{F}_q$. Then $\mathcal{H}^{\square}_{DAHA}=$ $\mathcal{H}^{mot}_{\mathcal{C}}\overset{\text{def}}{=\!=}\sum_{\{M_0\subset\cdots\subset M_\ell\}\in\mathcal{F}(\mathbb{F})} t^{\dim(\mathcal{O}/M_\ell)}a^\ell$ for the DAHA superpolynomial corresponding to $\mathcal{C}$; $\mathcal{H}^{mot}_{\mathcal{C}}$ generalize p-adic orbital integrals (type A, nil-elliptic), which are for $t=1, a=0$.

This is checked very well, incl. many cases with "cells" in $\mathcal{F}=\cup_{\mathcal{D}}\mathcal{F}_{\mathcal{D}}$ that are *not* **$\mathbb{A}^N$; here $\mathcal{D}\overset{\text{def}}{=\!=}\{D_0\subset\cdots\subset D_m\}$ for modules $D_i\overset{\text{def}}{=\!=}$valuation$_z(M_i)$ over the** *semigroup* **$\Gamma\overset{\text{def}}{=\!=}$valuation$_z(\mathcal{R})$.**

EXAMPLES OF PLAIN SINGULARITIES

The simplest one is for "trefoil" $T(3,2)$. The corresponding ring of singularity $\mathcal{R}=\mathbb{C}[\![z^2,z^3]\!]$ has the valuation semigroup $\Gamma=\mathbb{Z}_+\setminus\{1\}$. The latter remains unchanged over any(!) $\mathbb{F}_q$. The modules are $M_\lambda=(1+\lambda z)$ (called invertibles) of dim $\mathcal{O}/M=1$, and $M=\mathcal{O}$ (2 generators; dim$=0$). The standard 1-flags are $\{M_\lambda\subset\mathcal{O}\}$ (of dim 0). Thus $\mathcal{H}^{mot}=1\,$(for $\mathcal{O}$) $+\,qt\,$(invertibles) $+\,aq\,$(for 1-flags).

The simplest non-torus one is the ring $\mathcal{R}=\mathbb{C}[\![z^4,z^6+z^7]\!]$, where $\Gamma=\mathbb{Z}_+\setminus$Gaps for Gaps$=\{1,2,3,5,7,9,11,15\}$, so $\delta=|$Gaps$|=8$. Here $(z^6+z^7)^2-(z^4)^3=2z^{13}+\ldots$ and $p=2$ is a place of bad reduction. This singularity can be also presented by $\mathcal{R}'=\mathbb{C}[\![z^4+z^5,z^6]\!]$, where the reduction is bad only at $p=3$. Thus it has no places of bad reduction; the same holds for any algebraic knots (higher dimensions?). Importantly, in contrast to torus knots, only 23 out of the 25 Γ-modules D come from some standard M (*the Piontkowski phenomenon*).

FLAGGED GALKIN-STÖHR ZETA

A flag of $\mathcal{R}$-*ideals* $\mathcal{M}=\{M_0\subset M_1\cdots\subset M_\ell\}$ **is called** *standardizable* **if** $\{z^{-m}M_i\}$ **becomes standard (as above) for** $m = min(\textbf{valuation}_z(M_\ell))$**. We set:** $\mathcal{Z}(q,t,a)\stackrel{\textbf{def}}{=\!=} \sum_{\mathcal{M}\subset\mathcal{R}} a^\ell t^{dim_{\mathbb{F}}(\mathcal{R}/M_\ell)}$, $\mathcal{L}(q,t,a)\stackrel{\textbf{def}}{=\!=}(1-t)\mathcal{Z}(q,t,a)$, **where the summation is over** *standardizable* **flags of ideals** $\mathcal{M}$**.**

Conjecture. **Setting** $\mathbf{H}^{\boldsymbol{mot}}(\boldsymbol{q},\boldsymbol{t},\boldsymbol{a})\stackrel{\textbf{def}}{=\!=}\boldsymbol{\mathcal{H}}^{\boldsymbol{mot}}(\boldsymbol{qt},\boldsymbol{t},\boldsymbol{a})$, $\mathbf{H}^{\boldsymbol{mot}}(\boldsymbol{q},\boldsymbol{t},\boldsymbol{a})=\boldsymbol{\mathcal{L}}(\boldsymbol{q},\boldsymbol{t},\boldsymbol{a})$, **and** $\mathbf{H}^{mot}(q,t,a=-\frac{1}{q}) = \mathcal{L}_{\mathbf{prncpl}}(q,t,a=0)$ (**summation over principle** $\mathcal{M}\subset\mathcal{R}$)**, where the latter "=" possibly holds for any Gorenstein** $\mathcal{R}$**.**

Conjecture is checked for many knots, including involved cases when $\mathcal{R}=\mathbb{F}[[z^6,z^8+z^9]]$ **for** $\ell=0,1$ **and** $\mathcal{R}=\mathbb{F}[[z^6,z^9+z^{10}]]$ **for** $\ell=0$**.**

DAHA and AFFINE FLAG VARIETY

Another definition of DAHA is via the $T \times \mathbb{C}^*$-equivariant K-theory of the affine flag variety $\mathcal{B}$; T is the maximal torus. Namely, $\mathcal{H}$ is essentially $K^{T\times\mathbb{C}^*}(\Lambda)$ for a certain canonical Lagrangian subspace $\Lambda \subset \mathcal{T}^*(\mathcal{B}\times\mathcal{B})$, i.e. it is the Grothendieck ring of the (derived) category of $T \times \mathbb{C}^*$-equivariant coherent sheaves on Λ. This approach potentially leads to the classification of irreducible representations of $\mathcal{H}$; Ginzburg-Kapranov-Vasserot, Garland-Grojnowski, Varagnolo-Vasserot, … .

There is a connection to Bezrukavnikov's and others' recent research on "double-affine theories". The action of the double affine Weyl group on cohomology of an affine Springer fiber (Yun) is very important here.

AFFINE SPRINGER FIBERS

We assume that $\gamma \in \mathfrak{g}(\mathbb{F}((x)))$ **for a semi-simple Lie algebra** $\mathfrak{g}$ **is** *nil-elliptic*, **i.e. it is the unibranch case: no split tori over a local field** $\mathbb{F}((x))$ **in the stabilizer of** γ**. The** *affine Springer fiber* $\mathcal{X}_\gamma$ **is then formed by the classes of** g **in the** *affine Grassmannian* $G(\mathbb{F}((x)))/G(\mathbb{F}[[x)]])$ **over** $\mathbb{F} = \mathbb{F}_q$ **such that** $Ad_g^{-1}(\gamma) \in \mathfrak{g}(\mathbb{F}[[x]])$**, where Lie**$(G) = \mathfrak{g}$**. In type** A**, it can be identified with our compactified Jacobian for the characteristic polynomial** $\chi_\gamma(x,y)=\det(\gamma - y\mathbf{1})$**, which is far from obvious** *a priori* **due to different roles of** x, y**. The** p**-adic orbital integrals count** $\mathbb{F}$**-points of the latter, which corresponds to** $a=0, t=1$ **in** $\mathcal{H}$**. Our flagged construction is some counterpart of** *parahoric Springer fibers* **(for partially full flags).**

Our approach requires no matrices. E.g., the coincidence of superpolynomials for torus knots $T(r,s)$ **and** $T(s,r)$ **is direct.**

SOME NT PERSPECTIVES: Periods of cusp forms Φ of weight $w \geq 12$, namely $\int_{\gamma[0,\imath\infty]} z^k\,\Phi_\chi(z)dz$, result in *p-adic measures* **(Mazur, Manin, Katz,..., eigenvarieties) for $0 \leq k \leq w$. For us: $\int_0^{\imath\infty}\{\cdot\}\Phi_\chi dz \rightsquigarrow \{\cdot\} =$ coinvariant, $z^k \rightsquigarrow P_\lambda$.**

THM (DAHA-Satake). DAHA coinvariants of level ℓ ($\ell = 1$ above) are 1-1 with theta-functions (generally Looijenga functions) of level ℓ. P_λ can be replaced by *global hypergeometric functions* **(which, for instance, gives an approach to refined A-polynomials).**

The whole $PSL(2,\mathbb{Q})$ is expected to act in some variant of DAHA, the elliptic "polynomial" representation of the *same* **$\mathcal{H}$ is important, and more. The analogy between the Alexander and Iwasawa polynomials (Barry Mazur) was extended to the refined theory (where there is really some connection); a link to Kubota-Leopold p-adic zeta is not impossible.**

MORE NT CONNECTIONS

1. Ell.Conf.Space ≈ Bun$_G(E)$ for $G = SL_N$, which is related to LP for E; also $\mathcal{HH}^{\mathbf{S}_N}$ (spherical DAHA) $\rightsquigarrow$ elliptic Hall algebras (Schiffmann, Vasserot) $\rightsquigarrow$ W-algebras. A link to LP is when $t=0$.

2. As $q^n = 1$, DAHA dramatically generalize and simplify Verlinde algebras. Thm: $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts in *rigid* $\mathcal{HH}$-modules ($N = 2$). Potentially DAHA can produce modular forms directly from links (or Seifert 3-folds). Thm: DAHA coinvariants $\leftrightarrow$ elliptic functions.

3. There is a counterpart of the theory of zeta-functions of (local) curve singularities, where the *spectral zeta-function* is calculated for Dirac operator of the *Schottky uniformization* of a curve. Using here p-adic Mumford-Tate uniformization, where the closed fiber is the rational curve associated with the singularity, presumably leads to some (similar) theory of superpolynomials.

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ZETA FUNCTIONS

CLASSICAL THEORIES

Weil's conjectures for smooth mnflds

Riemann's zeta, Dirichlet L-fncts

No connection between these theories

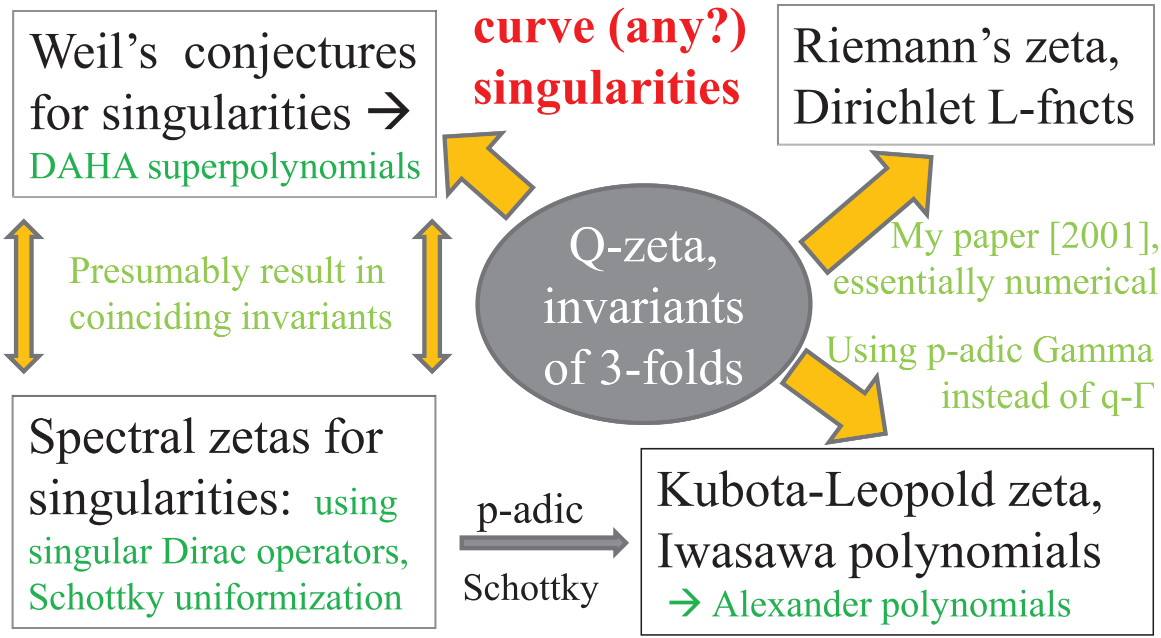
Spectral zetas, Selberg's zeta

Kubota-Leopold ζ Iwasawa polynms

For instance, Dirichlet L-functions have no counterparts among Weil's L-functions (and they have no q) : two different universes. Also, ***zeta-equivalence*** of algebraic varieties over $\mathbb{C}$ (N. Katz) generally results only in the coincidence of their Hodge numbers.

FOCUS ON SINGULARITIES

These theories become connected for curve (any?) singularities



If they capture the topological (!) invariants of links or 3-folds, then these theories must be ***a priori*** equivalent!

ZETAS AS TOPOLOGICAL INVARIANTS

First, we check that an isolated hypersurface singularity $0 \in \mathcal{X} \subset \mathbb{C}^n$ within its topological type, (say, the isotopy class of $\mathcal{X} \cap \mathbb{S}^{2n-1}$) can be defined over $\mathbb{Z}$ and generic (any?) $\mathbb{F}_q$, i.e. has good reductions at general prime p. Second, the compactified Jacobian $\bar{J}(\mathcal{X})$ of $\mathcal{X}$ (more generally, $Bun_G(\mathcal{X})$) is assumed of *strong polynomial count*: **$|\bar{J}(\mathcal{X})(\mathbb{F}_q)|$ depend polynomially on q. Then the** *flagged* **$\zeta_{\mathcal{X}}(q,t,a)$ is a powerful topological invariant of $\mathcal{X}$. For instance, $\zeta_{\mathcal{C}}(q,t,a)$ for a plane curve singularity $\mathcal{C}$, readily provides the valuation semi-ring of $\mathcal{C}$, which determines the topological type of unibranch $\mathcal{C}$. Similarly, the singular and p-adic zetas are expected to capture the topological type of $\mathcal{C}$ too, so we expect some a priori equivalence of these 3 theories. The passage to the "Grand" ζ, L-functions is (hopefully) via toric-type surface singularities $\mathcal{X}$ (and Seifert 3-folds).**