Highest weight categories and blocks in $\mathcal{O}$

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The blocks in BGG category $\mathcal{O}$

Highest weight categories and uniqueness

The ind-completion

BGG Category $\mathcal{O}$

Fix a reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.
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Then $\mathcal{O}(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}, \mathfrak{b})$ is the category of all $U(\mathfrak{g})$-modules which are

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Our choice of positive roots generates partial order $\leq$ on $\mathfrak{h}^*$.
Simple modules $\{L(\lambda)\}$ are labelled by their highest weight $\lambda \in \mathfrak{h}^*$.

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$$\mathcal{O}(\mathfrak{g}, \mathfrak{b}) \;=\; \bigoplus_{\lambda \in X^+} \mathcal{O}_\lambda(\mathfrak{g}, \mathfrak{b}).$$

Here, $X^+ \subset \mathfrak{h}^*$ are the dominant weights.

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Then $\mathcal{O}_\lambda(\mathfrak{g})$ has finitely many simples (labelled by $W(\lambda) \cdot \lambda$) and enough projective and injective objects.

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  - We only need to worry about integral blocks.
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  This means
  - We only need to worry about integral blocks.
  - Depending on your preference, feel free to ignore either $B_n = \mathfrak{so}(2n+1)$ or $C_n = \mathfrak{sp}(2n)$.
  - We can introduce the notation $\mathcal{O}(W, W')$ for a finite Weyl group and a parabolic subgroup $W' < W$.

Naive question: Does $\mathcal{O}(W, W') \simeq \mathcal{O}(U, U')$ imply there exists $\phi : W \xrightarrow{\sim} U$ as Coxeter groups with $\phi(W') = U'$?

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- **Boe, Nakano, Stroppel, ...**
  - $\mathcal{O}(A_{2n+1}, A_{2n}) \simeq \mathcal{O}(B_{n+1}, B_n)$, with $n \geq 2$;
  - $\mathcal{O}(B_n, A_{n-1}) \simeq \mathcal{O}(D_{n+1}, A_n)$, with $n \geq 3$;
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Any more? (Have we classified blocks up to equivalence?)

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Claim (see part II): The second property implies that $\mathcal{O}(W, W')$ can only admit one highest weight structure (up to 'equivalence').
$\Rightarrow$ Any equivalence

$$\mathcal{O}(W, W') \xrightarrow{\sim} \mathcal{O}(U, U')$$

must be equivalence of highest weight categories

Theorem (BGG)

Fix $\mathcal{O}(W,W')$. There is a non-zero morphism $\Delta(x) \to \Delta(y)$ for $x, y \in W/W'$ if and only if $y \preceq x$ in the Bruhat order on W/W'.

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Corollary

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$\mathcal{O}(W, W') \simeq \mathcal{O}(U, U')$ implies that $(W/W', \preceq) \simeq (U/U', \preceq)$.

Question: How much of (W, W') can we recover from $(W/W', \preceq)$?

Lemma (K.C.)

If for finite Weyl groups we have $(W/W', \preceq) \simeq (U/U', \preceq)$, then the pairs (W, W') and (U, U') are 'linked' by

- $(G, G) \leftrightarrow (H, H)$;
- $(A_{2n+1}, A_{2n}) \leftrightarrow (B_{n+1}, B_n)$, *with* $n \geq 2$;
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For the irreducible Weyl group case, the only non-trivial equivalences are

- $\mathcal{O}(G,G) \simeq \mathcal{O}(H,H)$;
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4. We have $[\Delta_1(\lambda)] = [\Delta_2(\lambda)]$ in $K_0(\mathbf{C})$, for all λ.

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Question. What about non-finite highest weight categories?

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Question:
Uniqueness in presence of simple preserving anti-autoequivalence?

Lemma (K.C.)

In a lower finite highest weight category with duality,

$$L(\lambda) = \Delta(\lambda) \quad \Leftrightarrow \quad \mathrm{Ext}^2(L(\lambda), L(\lambda)) = 0$$
$$\Leftrightarrow \quad \mathrm{Ext}^\bullet(L(\lambda), L(\lambda)) = k$$

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$\to$ Weyl modules in $\mathrm{Rep}\, G$ were not invented, but discovered

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Theorem (K.C.-V.Mazorchuk, K.C.)

The following conditions on an abelian category $\mathbf{A}$ *with abelian subcategory* $\mathbf{B}$ *are equivalent.*

1. *For every epimorphism* $p : A \twoheadrightarrow B$ *in* $\mathbf{A}$*, with* $B \in \mathbf{B}$*, there exists a morphism* $f : B' \to A$ *with* $B' \in \mathbf{B}$ *such that* $p \circ f$ *is still an epimorphism.*

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2. *For each* $B \in \mathbf{B}$ *and* $X \in \mathbf{A}$*, we get an epimorphism*

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3. *For each* $B \in \mathbf{B}$*,* $X \in \mathbf{A}$*, we get an isomorphism*
$$\mathrm{colim}_{Y \in \mathbf{B}/X} \mathrm{Ext}^i_{\mathbf{B}}(B, Y) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathbf{A}}(B, X), \qquad \forall i \in \mathbb{N}.$$

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We do **not** have

$$\varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Ext}^i_{\mathbf{C}}(X_\beta, Y_\alpha) \simeq \mathrm{Ext}^i_{\mathrm{Ind}\mathbf{C}}(\varinjlim_{\beta} X_\beta, \varinjlim_{\alpha} Y_\alpha),$$

for $i > 0$.