Symplectic leaves in Calogero–Moser spaces

Ulrich Thiel

TU Kaiserslautern
https://ulthiel.com/math

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Basic rationale of the talk

1. Every symplectic manifold is naturally a Poisson manifold.
2. A Poisson manifold can be stratified into symplectic submanifolds: the symplectic leaves.
3. Under some assumptions, this works for (possibly singular) Poisson algebraic varieties as well.
4. Goal: describe symplectic leaves of Poisson deformations (Calogero–Moser spaces) of symplectic linear quotients V/Γ, $\Gamma \subset \mathrm{Sp}(V)$ finite.
5. Motivation: involves interesting representation theory, several open problems and conjectures.

Analytic theory

Poisson manifolds

Definition

A Poisson structure on an algebra A is a Lie bracket $\{\cdot, \cdot\}$ on A which satisfies the Leibniz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

A Poisson manifold is a manifold M with a Poisson structure on $C^\infty(M)$.

Example (Lie, 1890)

$\mathfrak{g}$ a Lie algebra. Then $\mathfrak{g}^*$ is a Poisson manifold with $\{f, g\} = [f, g]$ for $f, g \in \mathfrak{g}$.

Key property

For any $f \in C^\infty(M)$, $\{f, -\}$ is a derivation on $C^\infty(M)$, hence defines a vector field X_f on M. Such vector fields are called Hamiltonian.

The Poisson bivector

Recall: X_f is the vector field associated to $\{f, -\}$.

By definition, $\{f, g\} = dg(X_f) = -df(X_g)$.

Hence, there is $\pi \in \Gamma(\wedge^2 TM)$ such that $\{f, g\} = \langle df \wedge dg, \pi \rangle$.

π is called the Poisson bi-vector.

π defines a map $\pi^\sharp \colon T^*M \to TM$ with $X_f = \pi^\sharp(df)$.

Definition

The rank of $\{\cdot, \cdot\}$ at $p \in M$ is the rank of $\pi^\sharp_p$.

Symplectic manifolds

Definition

A symplectic form on a manifold M is a closed non-degenerate 2-form ω on M.

Example

$\mathfrak{h}$ a vector space, then $\mathfrak{h} \oplus \mathfrak{h}^*$ is symplectic with

$$\omega((y,x),(y',x')) := x'(y) - x(y') \ .$$

Key property

ω induces $\omega^\flat : TM \xrightarrow{\sim} T^*M$.

Hence, every $f \in C^\infty(M)$ defines a vector field $X_f = (\omega^\flat)^{-1}(df)$, i.e. $df(Y) = \omega(X_f, Y)$.

This yields a Poisson structure $\{f, g\} := \omega(X_f, X_g)$.

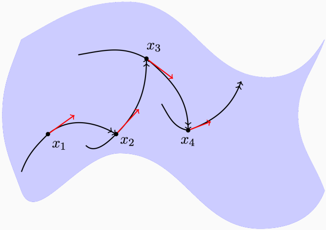
For this structure, $\pi^\sharp = -(\omega^\flat)^{-1}$.

Hence: Poisson structures of full rank = symplectic structures.

Symplectic leaves

Definition

An integral curve of a vector field X is a curve γ on M such that $X(\gamma(t)) = \gamma'(t)\ \forall t$.



Definition (Weinstein, 1983)

M a Poisson manifold. The symplectic leaf $\mathcal{L}(p)$ through $p \in M$ is the set of all $q \in M$ which are connected to p by piecewise smooth curves, each segment being the integral curve of a Hamiltonian vector field.

Symplectic leaves

Clearly, the leaves partition M.

Fact

Each leaf is a Poisson submanifold on which the Poisson structure is of full rank, i.e. it is symplectic.

Example (Kostant–Kirillov–Souriau, 1960s)

$\mathfrak{g} = \mathrm{Lie}(G)$, then the symplectic leaves of $\mathfrak{g}^*$ are the G-orbits in $\mathfrak{g}^*$ (coadjoint orbits).

Algebraic theory

Poisson and symplectic varieties

X: affine algebraic variety $/\mathbb{C}$, not necessarily smooth.

Definition

- A Poisson structure on X is a Poisson structure on $\mathbb{C}[X]$.
- If X is smooth, a symplectic form on X is a closed non-degenerate 2-form on X.

Note: if X is smooth and Poisson, so is X^{an} (use Poisson bi-vector).

Example

(V, ω) a symplectic vector space. This is a smooth symplectic variety.

Induces a Poisson structure $\{\cdot, \cdot\}$ on $\mathbb{C}[X]$.

If $\Gamma \subset \mathsf{Sp}(V)$ is a finite subgroup, then $\{\cdot, \cdot\}$ is G-invariant, hence $\{\cdot, \cdot\}$ descends to $\mathbb{C}[V]^\Gamma$, hence $V/\Gamma = \mathsf{Spec}\,\mathbb{C}[V]^\Gamma$ is Poisson.

Symplectic leaves (Brown–Gordon)

X: a (not necessarily smooth) Poisson variety.

The smooth locus U_0 is a smooth Poisson variety, hence U_0^{an} is a smooth Poisson manifold, hence has a stratification

$$U_0^{an} = \coprod_{i \in I_0} \mathcal{L}_{0,i}$$

into symplectic leaves.

Now, proceed inductively with $X_1 := X \setminus U_0$ (this is Poisson).

We get a stratification

$$X = \coprod_{\substack{k \in \mathbb{N} \\ i \in I_k}} \mathcal{L}_{k,i} \ .$$

The $\mathcal{L}_{k,i}$ are the symplectic leaves of X.

Symplectic leaves (Brown–Gordon)

Problem

A priori, there is no reason why the leaves are algebraic, i.e. locally closed subvarieties.

Facts

If there are only finitely many leaves, then they are algebraic.

The leaf of p is the set of all q having the same Poisson core as p.

Each leaf $\mathcal{L}$ is irreducible and $\mathcal{L} = \overline{\mathcal{L}}^{sm}$.

Example

Symplectic leaves of $V/\Gamma \overset{1:1}{\longleftrightarrow}$ conjugacy classes of parabolic subgroups of Γ (point-wise stabilizer of a subspace of V)

$$V/\Gamma = \coprod_{(P)} \pi(V_P) \;, \quad \pi : V \to V/\Gamma \;.$$

Symplectic leaves in Calogero–Moser spaces

Calogero–Moser spaces (Etingof–Ginzburg)

$W \subset \mathsf{GL}(\mathfrak{h})$ a complex reflection group.

With induced action on $\mathfrak{h}^*$, have $W \subset \mathsf{Sp}(\mathfrak{h} \oplus \mathfrak{h}^*)$, so $X \coloneqq (\mathfrak{h} \oplus \mathfrak{h}^*)/W$ special case of V/Γ as before.

Poisson deformations of X can be described as follows.

Choose parameters: $c \in \mathbb{C}^{\mathsf{Refl}(W)/W}$ and $t \in \mathbb{C}$.

Let $H_{t,c}$ be the quotient of $\mathbb{C}\langle \mathfrak{h} \oplus \mathfrak{h}^* \rangle \rtimes W$ by

$$[x, x'] = 0 = [y, y'] \quad \forall x, x' \in \mathfrak{h}^*,\ y, y' \in \mathfrak{h}$$

$$[x, y] = t\langle x, y\rangle - \sum_{s \in \mathsf{Refl}(W)} c(s) \frac{\langle y, \alpha_s\rangle\langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} s \quad \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}$$

$H_{t,c}$ is the rational Cherednik algebra.

Calogero–Moser spaces (Etingof–Ginzburg)

Facts

1. $H_{t,c}$ is flat over $\mathbb{C} \times \mathbb{C}^{\mathrm{Refl}(W)/W}$.
2. $Z_c := Z(H_{0,c})$ is a normal finite type $\mathbb{C}$-algebra. $X_c := \mathrm{Spec}(Z_c)$ is called Calogero–Moser space.
3. $Z_0 = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$, so $X_0 = (\mathfrak{h} \oplus \mathfrak{h}^*)/W = X$.
4. Think of $H_{0,c}$ as $H_{t,c}/tH_{t,c}$. For $z_1, z_2 \in Z_c$ define
$$\{z_1, z_2\} := \left(\frac{1}{t}[\hat{z}_1, \hat{z}_2]\right) \bmod tH_{t,c} \ .$$
This is a Poisson structure on Z_c.
5. The X_c give all the Poisson deformations of X (Bellamy, Losev).

Task

Describe the symplectic leaves of X_c for all W, c.

Brown–Gordon

X_c has only finitely many leaves, hence they are algebraic.

Leaves and parabolic subgroups (Bellamy)

Case $c = 0$

Leaves of $(\mathfrak{h} \oplus \mathfrak{h}^*)/W \xleftrightarrow{1:1}$ conj. cl. of parabolic subgroups of W.

Smooth case

If X_c is smooth, it is the unique symplectic leaf.

This holds precisely for $G(l, 1, n) = C_l \wr S_n$ and G_4 for generic c.

In general

We have inclusions $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow Z_c$ and $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_c$. Hence, morphisms

$$\pi_c : X_c \twoheadrightarrow \mathfrak{h}/W \ , \quad \varpi_c : X_c \twoheadrightarrow \mathfrak{h}^*/W \ , \quad \Upsilon_c : X_c \twoheadrightarrow \mathfrak{h}/W \times \mathfrak{h}^*/W \ .$$

In fact, Υ_c is finite. So, $\Upsilon_c(\mathcal{L})$ locally closed, $\dim \Upsilon_c(\mathcal{L}) = \dim \mathcal{L} = 2m$.

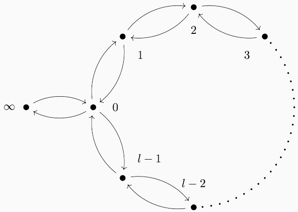
$\Upsilon_c(\mathcal{L}) \subset \pi_c(\mathcal{L}) \times \varpi_c(\mathcal{L})$, so $\dim \pi_c(\mathcal{L}) \geq m$ or $\dim \varpi_c(\mathcal{L}) \geq m$.

There is a unique conjugacy class $W_{\mathcal{L}} := (W')$ of parabolic subgroups such that $\pi_c(\mathcal{L}) \cap \mathfrak{h}_{reg}^{(W')}/W$ is dense in $\pi_c(\mathcal{L})$.

Leaves via quiver varieties (Etingof–Ginzburg, Martino)

Fact

For $G(l,1,n)$, X_c is Poisson isomorphic to a quiver variety $\mathcal{X}_{\theta_c}(d)$ for



Symplectic leaves are identified with the representation type strata.

Type B 

$c = (\kappa, c_1)$

- X_c is singular iff $\kappa = 0$ or $c_1 = m\kappa$ with $m \in \pm\{0, \dots, n-1\}$.
- 1st case: leaves parametrized by $\mathcal{P}(n)$ and $\dim \mathcal{L}_\lambda = 2\ell(\lambda)$.
- 2nd case: leaves parametrized by $\{k \in \mathbb{N} \mid k(k+m) \leq n\}$ and $\dim \mathcal{L}_k = 2(n-(k+m))$.

“Clifford theory” for leaves (Bellamy–T.)

$K \trianglelefteq W$, acting on $\mathfrak{h}$ as a reflection group, $\Gamma \coloneqq W/K$.

$c\colon \operatorname{Refl}(K) \to \mathbb{C}$ a W-equivariant function, extend to $\operatorname{Refl}(W) \to \mathbb{C}$ by 0.

$H_{0,c}(K) \hookrightarrow H_{0,c}(W)$, $Z_c(W) = Z_c(K)^\Gamma$, $Z_c(W) \hookrightarrow Z_c(K)$ is Poisson.

Poisson morphism $\eta\colon X_c(K) \to X_c(W)$ identifying $X_c(W)$ with $X_c(K)/\Gamma$.

Type D

$D_n \triangleleft B_n$, $\Gamma = C_2$. With $c = (0, \kappa)$ we are in situation as above.

For $\kappa \neq 0$ the leaves of $X_\kappa(D_n)$ are parametrized by $\{k \geq 1 | k^2 \leq n\}$ and $\dim \mathcal{L}_k = 2(n - k^2)$.

Results and conjectures (Bellamy–T.)

We know the leaves for all non-exceptional finite Coxeter groups.

Conjectures

For any finite Coxeter group W and any c:

1. Each conjugacy class of parabolics labels at most one leaf.
2. The geometric ordering on leaves ($\mathcal{L} \leq \mathcal{L}'$ if $\mathcal{L} \subseteq \overline{\mathcal{L}}'$) equals the algebraic ordering ($(W_1) \leq (W_2)$ if W_2 is conjugate to a subgroup of W_1).
3. There is at most one zero-dimensional leaf.

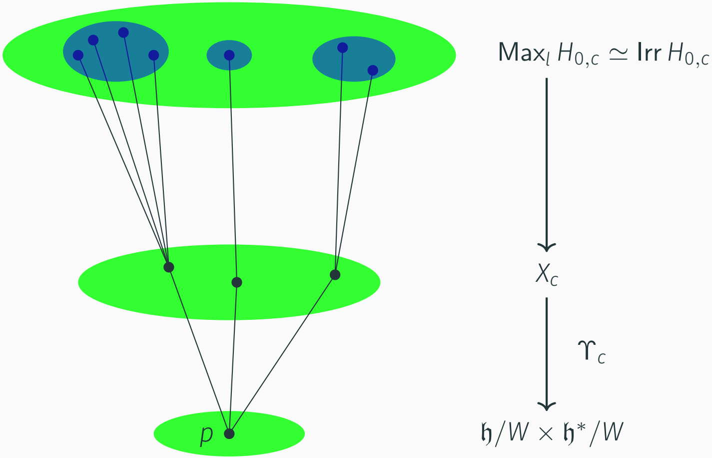
Note: All wrong in general for complex reflection groups.

Work in progress

Leaves for $G(l, 1, n)$, and then $G(l, p, n)$.

Representation theory associated to symplectic leaves

Geometry and representation theory



$\Upsilon_c^{-1}(p) \simeq$ blocks of $H_{0,c}/\mathfrak{m}_p H_{0,c}$.

Zero-dimensional leaves

Losev

To determine the leaves, it is basically sufficient to determine the zero-dimensional leaves for all parabolic subgroups.

Fact

All zero-dimensional leaves of X_c are contained in $\Upsilon_c^{-1}(0) = X_c^{\mathbb{C}^*}$.

Hence, they correspond to certain blocks of

$$\overline{H}_c := H_{0,c}/\mathfrak{m}_0 H_{0,c}$$

There is a natural bijection (Gordon)

$$\operatorname{Irr} W \simeq \operatorname{Irr} \overline{H}_c \ , \quad \lambda \mapsto L_c(\lambda) \ .$$

Hence, zero-dimensional leaves $\leftrightarrow$ certain families of W-characters.

Call such families cuspidal.

Question

What are the cuspidal families?

Cuspidal = not induced (Bellamy)

Fact

If $L \in \operatorname{Irr} H_{0,c}$ lies on a leaf $\mathcal{L}$ with $\dim \mathcal{L} > 0$, then L is induced: there is a proper parabolic $W' \subsetneq W$ such that $L \simeq \operatorname{Ind}_{W'}^{W} M$ as W-modules for some W'-module M.

Corollary

If L is not induced, it lies on a zero-dimensional leaf.

Problem

We don't know much about the structure of simple $\overline{H}_c$-modules.

Rigid representations (Bellamy–T.)

Definition

Say that $L \in \operatorname{Irr} H_{0,c}$ is rigid if it is irreducible as a W-module.

Remark

If L is rigid of dimension d, the isomorphism class of L in the representation scheme $\mathsf{Rep}_d(H_{0,c})$ is a connected component.

Theorem

No irreducible character of a complex reflection group W is induced from a proper parabolic subgroup.

Our proof: brute-force using branching rules for $G(l, 1, n)$, Clifford theory for $G(l, p, n)$, and computer for exceptionals.

S. Griffeth later sent us a general argument!

Corollary

If L is rigid, it lies on a zero-dimensional leaf. Call such leaves rigid.

Identification of cuspidal families (Bellamy–T.)

To find rigid representations, just need to check whether $\lambda \in \mathrm{Irr}(W)$ lifts to $H_{0,c}$, i.e. if it satisfies the relations. That's easy.

Theorem

For all non-exceptional finite Coxeter groups, the zero-dimensional leaves are rigid.

Surprising fact (Gordon, Gordon–Martino, Bellamy, T., Bonnafé–T.)

For any finite Coxeter group (except possibly H_4 and Es) and any c the families of W-characters defined by $\overline{H}_c$ equal Lusztig's families of unipotent characters (for the same c).

Theorem

For all non-exceptional finite Coxeter groups, the cuspidal families of $\overline{H}_c$ equal Lusztig's cuspidal families (not $\mathbf{j}$-induced).

Conjecture

This is true for all finite Coxeter groups.
D. Ciubotaru: approach via Dirac cohomology. E_7 is interesting!

Hyperplanes (Bellamy–Schedler–T.)

W: a complex reflection group.

Let $N := \max_c \#\{\text{blocks of } \overline{H}_c\} < \infty$.

Let $\mathcal{E} := \{c \mid \#\{\text{blocks of } \overline{H}_c\} < N\}$.

Theorem

$\mathcal{E}$ is an arrangement of finitely many hyperplanes (because $\mathcal{E}$ is equal to Namikawa's arrangement parametrizing $\mathbb{Q}$-factorial terminalizations of $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$).

Easy fact

If $X_{\mathbf{c}}$ is smooth for some $\mathbf{c}$, then X_c is smooth iff $c \notin \mathcal{E}$.

Conjecture

In general, $\mathcal{E}$ controls the symplectic leaves (i.e., their parametrization and dimensions).